

CONVEXITY AND CONCAVITY OF THE GROUND STATE ENERGY

HERBERT KOCH

ABSTRACT. This note proves convexity resp. concavity of the ground state energy of one dimensional Schrödinger operators as a function of an endpoint of the interval for convex resp. concave potentials.

1. MAIN RESULT AND CONTEXT

Let $I = (a, b) \subset \mathbb{R}$ be an open interval, $V \in C(a, b)$ be a convex or concave potential with $\liminf_{t \rightarrow -\infty} V = \infty$ if $a = -\infty$. Consider for $t \in (a, b]$ the energy

$$E_t(u) = \int_a^t u_x^2 + V u^2 dx.$$

There is a unique positive minimizer $u \in H_0^1(a, t)$ under the constraint $\|u\|_{L^2(a, t)} = 1$. It satisfies the Euler-Lagrange equation

$$(1) \quad -u_{xx} + V u = \lambda(t) u$$

on (a, t) with boundary conditions $u(a) = u(t) = 0$ (and obvious modifications if $a = -\infty$). Here $\lambda(t)$ is the Lagrangian multiplier, and $\lambda(t) = E_t(u)$. The map $t \rightarrow \lambda(t)$ is the main object of interest.

Theorem 1. *The map $(a, b] \ni t \rightarrow \lambda(t)$ is twice differentiable, strictly decreasing and $\lim_{t \rightarrow a} \lambda(t) = \infty$. The map $t \rightarrow \lambda(t)$ is convex if V is convex, strictly convex if V is convex and not affine. If $a = -\infty$ it is concave if V is concave and strictly concave if V is concave and not affine.*

The convexity part follows from a much stronger celebrated result by Brascamp and Lieb [3, 4]. It is related to a weaker statement in Friedland and Hayman [6] with a computer based proof there. These statements found considerable interest and use in the context of monotonicity formulas beginning with the seminal work of Alt, Caffarelli and Friedman [1]. Caffarelli and Kenig [5] prove a related monotonicity formula using the results by Brascamp-Lieb [3]. They attribute an analytic proof to Beckner, Kenig and Pipher [2] which the author has never seen. To the best knowledge of the author the concavity statements are new.

1991 *Mathematics Subject Classification.* 34B09.

Key words and phrases. Ground state energy, convexity.

This note has its origin in a seminar of free boundary problems at Bonn. It is a pleasure to acknowledge that it would not exist without my coorganizer Wenhui Shi. I am grateful to Elliott Lieb for spotting an error in the formulation of the main theorem in a previous version.

2. A SHORT ELEMENTARY PROOF

Proof. Monotonicity and $\lim_{t \rightarrow a} \lambda(t) = \infty$ are an immediate consequence of the definition. We consider the equation (1) on the interval (a, t) and denote by $u(x) = u(x, t)$ the unique L^2 normalized non negative ground state with ground state energy $\lambda = \lambda(t)$. Differentiability with respect to x and t is an elementary property of ordinary differential equations. We argue at a formal level and do not check existence of integrals resp. derivatives below, which follows from standard arguments. We differentiate the equation with respect to t , denote the derivative of with respect to t by \dot{u} and obtain

$$(2) \quad -\dot{u}_{xx} + V\dot{u} - \lambda\dot{u} = \dot{\lambda}u$$

with boundary conditions $\dot{u}(a) = 0$ and $\dot{u}(t) = -u_x(t)$. We multiply (2) by u , integrate and integrate by parts. Then most terms drop out by (1). Since $\|u\|_{L^2} = 1$ we obtain

$$(3) \quad \dot{\lambda} = \dot{u}(t)u_x(t) = -u_x^2(t).$$

Due to the normalization \dot{u} is orthogonal to u , i.e. $\int_a^t u\dot{u}dx = 0$. The quotient $w = \frac{\dot{u}}{u}$ satisfies

$$w_{xx} + \frac{u_x}{u}w_x - \frac{u_x^2}{u^2}w = \dot{\lambda} < 0.$$

In particular w has no non positive local minimum. Since $w \rightarrow \infty$ as $x \rightarrow t$ there can be at most one sign change. Since \dot{u} is orthogonal to u there is exactly one sign change of \dot{u} , let's say at $a < t_0 < t$. Since also $\dot{u}(a) = 0$ if $a > -\infty$ we have $\dot{u}_x(a) \leq 0$ if $a > -\infty$. We multiply (1) by u_x and integrate to get

$$(4) \quad \dot{\lambda} = -u_x(t)^2 = \int_a^t V'u^2dx - u_x^2(a)$$

where we omit the last term here and below if $a = -\infty$.

We differentiate (4) with respect to t and use the orthogonality $\int_a^t u\dot{u}dx = 0$ to obtain a partly implicit formula for the second derivative of λ with respect to t ,

$$\begin{aligned} \ddot{\lambda} &= 2 \int_a^t (V'(x) - V'(t_0))u\dot{u}dx - 2u_x(a)\dot{u}_x(a) \\ &= 2 \int_a^t (V'(x) - V'(t_0))wu^2dx - 2u_x(a)\dot{u}_x(a). \end{aligned}$$

Recall that $u_x(a) > 0$ and $\dot{u}_x(a) \leq 0$ and hence the second term on the right hand side is nonnegative. By the choice of t_0 the first term is nonnegative

if V is convex, nonpositive if it is concave, positive if V is convex and non affine, and negative if V is concave and not affine. Thus $t \rightarrow \lambda$ is convex if V is convex, it satisfies $\ddot{\lambda} > 0$ if V is convex and not affine (i.e. V' is not constant), if $a = -\infty$ it is concave if V is concave and $\ddot{\lambda} < 0$ if V is concave and not affine. \square

REFERENCES

- [1] Alt, H.-W., Caffarelli, L. A. and Friedman, A.. Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc.*, 282(2):431–461, 1984.
- [2] Beckner, W., Kenig C. E., and Pipher, J.. A convexity property for Gaussian measures. 1998.
- [3] Brascamp, H. J. and Lieb, E. H.. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.
- [4] Brascamp, H.J. and Lieb, E. H.. Some Inequalities for Gaussian Measures and the long-range order if the one-dimensional plasma. In *Functional Integration and its Applications*, A.M. Arthurs ed., Clarendon Press, 1-14 (1975). Reprinted in M. Loss and M.B. Ruskai: *Inequalities*, Springer, 403–416 (2002).
- [5] Caffarelli, L. A. and Kenig, C. E.. Gradient estimates for variable coefficient parabolic equations and singular perturbation problems. *Amer. J. Math.*, 120(2):391–439, 1998.
- [6] Friedland, S. and Hayman, W. K.. Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions. *Comment. Math. Helv.*, 51(2):133–161, 1976.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN

E-mail address: koch@math.uni-bonn.de